

# Can we measure structures to a precision better than the Planck length?

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## Abstract.

It was recently claimed that the Planck length is not a limit to the precision by which we can measure distances, but that instead it is merely the Planck volume that limits the precision by which we can measure volumes. Here, we investigate this claim and show that the argument does not support the conclusion.

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## 1. Introduction

Since almost 80 years now, evidence has mounted that the Planck length,  $l_{\text{Pl}}$ , plays the rôle of a minimal length or, in other words, that it sets a limit to how precisely we can measure structures. Without an experimentally verified theory of quantum gravity, the existence of a minimal length scale is an expectation rather than a knowledge, yet the expectation that the bound

$$\Delta x^\nu \gtrsim l_{\text{Pl}} \quad , \quad (1)$$

is obeyed for spatial and temporal extensions has been supported by many thought experiments and different approaches to quantum gravity [1, 2].

Surprisingly, it was recently claimed by Tomassini and Viaggiu [3], building up on an earlier heuristic argument [4], that the Planck scale does not constitute a limit to the precision by which we can measure distances, but that instead we merely have a limit to the precision by which space-time volumes can be measured

$$\Delta x^0 (\Delta x^1 + \Delta x^2 + \Delta x^3) \gtrsim l_{\text{Pl}}^2 \quad . \quad (2)$$

Since the inequality (2) follows from (1) the relevant question here is not whether (2) is valid, but whether (1) can be violated.

That spatial distances can be measured to a precision better than the Planck length is an extraordinary claim which, if correct, would mean nothing less than that arguments dating back to Bronstein's 1936 paper [5] are all wrong. It would also pose a serious conceptual challenge to approaches to quantum gravity which have shown indications for

a minimal length scale, such as loop quantum gravity and asymptotically safe gravity. It is the aim of this paper to investigate Tomassini and Viaggiu's (in the following referred to as TV) argument and we will find it wanting. However, we wish to make this a constructive criticism, and more important than pointing out why the argument is on shaky ground, we want to clarify which steps are missing to put it on solid ground.

This paper is organized as follows. In the next section, we recall Mead's argument for the existence of a minimal length scale from studying the Heisenberg microscope by taking into account general relativity. It is one of the most general and also most convincing arguments. In section 3 we summarize the core of TV's argument. In section 4, we explain the differences in the argumentation, why the case for the absence of a minimal length does not hold up to scrutiny. We conclude in section 5.

We use the unit convention  $c = \hbar = 1$ , so that the Planck length  $l_{\text{Pl}}$  is the inverse of the Planck mass,  $m_{\text{Pl}} = 1/l_{\text{Pl}}$ , and Newton's constant  $G = l_{\text{Pl}}^2$ .

## 2. The case for a minimal length

Let us first recall Heisenberg's microscope, that lead to the uncertainty principle [7]. Consider a photon with frequency  $\omega$  moving in direction  $x$  which scatters on a particle whose position on the  $x$ -axis we want to measure. The scattered photons that reach the lens of the microscope have to lie within an angle  $\epsilon$  to produces an image from which we want to infer the position of the particle (see figure 1). According to classical optics, the wavelength of the photon sets a limit to the possible resolution  $\Delta x$

$$\Delta x \gtrsim \frac{1}{\omega \sin \epsilon} \gtrsim \frac{1}{\omega} \quad . \quad (3)$$

(Here and in the following we omit factors of order one; they do not matter for our argument.) But the photon used to measure the position of the particle has a recoil when it scatters and transfers a momentum to the particle. Since one does not know the direction of the photon to better than  $\epsilon$ , this results in an uncertainty for the momentum of the particle in direction  $x$

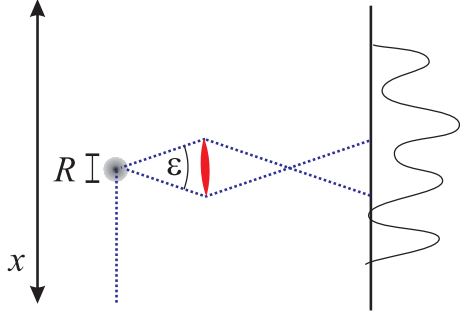
$$\Delta p_x \gtrsim \omega \sin \epsilon \quad . \quad (4)$$

Taken together one obtains, up to a factor of order one, Heisenberg's uncertainty

$$\Delta x \Delta p_x \gtrsim 1 \quad . \quad (5)$$

We know today that Heisenberg's uncertainty is much more than a peculiarity of microscopy; it is a fundamental principle of quantum mechanics. It does strictly speaking not even make sense to speak of the position and momentum of the particle at the same time. Consequently, instead of speaking about the photon scattering of the particle as if that would happen in one particular point, we should speak of the photon having a strong interaction with the particle in some region of size  $R$ .

Now we include gravity into the picture, following the treatment of Mead [6]. As before, we have a particle whose position we want to measure by help of a test particle.



**Figure 1.** Heisenberg's microscope.

For any interaction to take place and subsequent measurement to be possible, the time elapsed between the interaction and measurement has to be at least of the order of the time,  $\tau$ , the test particle needs to travel the distance  $R$ , so that  $\tau \gtrsim R$ . The test particle carries an energy that, though normally tiny, exerts a gravitational pull on the particle whose position we wish to measure. It is this gravitational pull exerted by the test particle, together with the limits by which we can know both its direction and momentum, that causes an additional uncertainty.

The test particle has a momentum vector  $(\omega, \vec{k})$ , and for completeness we consider a particle with rest mass  $\mu$ , though we will see later that the tightest constraints come from the limit  $\mu \rightarrow 0$ . The velocity  $v$  of the test particle is

$$v = \frac{k}{\sqrt{\mu^2 + k^2}} \quad , \quad (6)$$

where  $k^2 = \omega^2 - \mu^2$  and  $k = |\vec{k}|$ . As before, the test particle moves into direction  $x$ . The task is now to compute the gravitational field of the test particle and the motion it causes for the measured particle.

To obtain the metric that the test particle creates, we first change into the rest frame of the particle by boosting into  $x$ -direction. Denoting the new coordinates with primes, the measured particle moves towards the test particle in direction  $-x'$ , and the metric is a Schwarzschild metric. We will only need it on the  $x$ -axis where we have  $y = z = 0$ , and thus

$$g'_{00} = 1 + 2\phi' \quad , \quad g'_{00} = -\frac{1}{g'_{00}} \quad , \quad g'_{22} = g'_{33} = -1 \quad , \quad (7)$$

where  $\phi' = -G\mu/|x'|$ , and the remaining components of the metric vanish. A Lorentz-boost back into the rest frame of the measured particle yields

$$g_{00} = \frac{1 + 2\phi}{1 + 2\phi(1 - v^2)} + 2\phi \quad , \quad g_{11} = -\frac{-1 + 2\phi v^2}{1 + 2\phi(1 - v^2)} + 2v^2\phi \quad (8)$$

$$g_{01} = g_{10} = -\frac{2v\phi}{1 + \phi(1 - v^2)} - 2v\phi \quad , \quad g'_{22} = g'_{33} = -1 \quad , \quad (9)$$

where

$$\phi = \frac{\phi'}{1 - v^2} = -\frac{G\omega}{R} \quad . \quad (10)$$

Here,  $R = vt - x$  is mean distance between test particle and measured particle. To avoid a horizon in rest frame, we must have  $2\phi' < 1$ , and thus from Eq. (10)

$$-2\phi' = 2\frac{G\omega}{R}(1 - v^2) < 1 \quad . \quad (11)$$

Because of Eq. (3),  $\Delta x \geq 1/\omega$  but also  $\Delta x \geq R$  which is the area in which the particle may scatter, and therefore

$$\Delta x^2 \gtrsim \frac{R}{\omega} \gtrsim 2G(1 - v^2) \quad . \quad (12)$$

Thus, as long as  $v^2 \ll 1$ , the previously found lower bound on the spatial resolution  $\Delta x$  can already be read off here, and we turn our attention towards the case where  $1 - v^2 \ll 1$ . From (10) we see that this means we work in the limit where  $-\phi \gg 1$ .

To proceed, we need to estimate now how much the measured particle moves due to the test particle's vicinity. We denote the velocity in  $x$ -direction be  $u$ , then the requirement that the line-element  $ds^2 > 0$  on the particle's worldline yields, after some algebra, with (9) the estimate

$$\frac{u}{1 - u} \geq -\frac{1}{2}(1 + 2\phi) \quad . \quad (13)$$

The time  $\tau$  required for the test particle to move a distance  $R$  away from the measured particle is at least  $\tau \gtrsim R/(1 - u)$ , and during this time the measured particle moves a distance

$$L = u\tau \gtrsim R\frac{u}{1 - u} \gtrsim \frac{R}{2}(-1 - 2\phi) \quad . \quad (14)$$

Since we work in the limit  $-\phi \gg 1$ , this means  $L \gtrsim G$ , and projection on the  $x$ -axis yields for the uncertainty added to the measured particle because the photon's direction was known only to precision  $\epsilon$

$$\Delta x \gtrsim G\omega \sin \epsilon \quad . \quad (15)$$

This additional uncertainty combines with (3) to a lower limit for the spatial uncertainty given by the Planck length

$$\Delta x \gtrsim l_{\text{Pl}} \quad . \quad (16)$$

Mead continues to show that by a similar argument one finds that the precision by which clocks can be synchronized, and thus time intervals can be measured, is also bound by the Planck length,  $\Delta x^0 \gtrsim l_{\text{Pl}}$ .

Adler and Santiago [8] find the same result by using the linear approximation of Einstein's field equation for a cylindrical source with length  $l$  and radius  $\rho$  of comparable size, filled by a radiation field with total energy  $\omega$ , and moving into direction  $x$ . However, this estimate can be criticized on the grounds that the weak field approximation is strictly speaking inappropriate.

Several other thought experiments can be found in the literature, for example Wigner and Salecker derived limits to the precision of time and length measurements

by studying Einstein's synchronization procedure with gravity [9]. Scardigli [10] offered a related argument from the creation and subsequent evaporation of Planck scale black holes. Noteworthy is also Calmet, Graesser and Hsu's argument [11] for non-relativistic masses, which has the merit of being device-independent. Limits to the measurements of black hole horizons themselves have been studied in [12, 13]. They all arrive, up to a factor of order one, at the same bounds. Ng and van Dam [14] argued that the scaling behavior might be different from the one discussed here, but the lower limits remain the same. For more details, the interested reader is referred to the recent review [2].

As before with the normal Heisenberg microscope, the relevance of (16) is not one for microscopy. The microscope is only a placeholder for any scattering process. In fact, at the energies needed to probe the Planck scale, the test particle almost certainly would not scatter elastically. Instead, we should imagine a fixed-target collider experiment, in which we try to test for a possible substructure below the Planck scale. The inequality (16) tells us that this is not possible. This then raises the question if not a quantum theory that takes into account gravity should have built in this finite position uncertainty, an idea that has received a lot of attention since Snyder [15] showed that it need not be in conflict with Lorentz invariance.

### 3. The case for a minimal volume

The observant reader will have noticed that the above estimate made use of spherical symmetry for the gravitational field of the test particle. Adler and Santiago [8] employed cylindrical symmetry; however, also there it was assumed that the length and the radius of the cylinder are of comparable size. In fact, all the other thought experiments that arrive at the conclusion that the Planck length sets a limit to spatial resolution make implicitly or explicitly use of spherical symmetry, in most cases by using a condition that the extension of a mass distribution be larger than its Schwarzschild radius.

In the general case however, when the dimensions of the test particle in different directions are very unequal, the Hoop conjecture does not forbid any one direction to be smaller than the Schwarzschild radius to prevent collapse of some matter distribution, as long as at least one other direction is larger than the Schwarzschild radius. Leaving aside that it is called a conjecture because it is unproven and just taking it at face value, the question then arises what limits on spatial resolution can we still derive in the general case.

A heuristic motivation of the following argument can be found in [4], but we follow here the more detailed argument by Tomassini and Viaggiu [3]. In the absence of spherical symmetry, one may still use Penrose's isoperimetric-type conjecture [16, 17], according to which the area of the apparent horizon is always smaller or equal than the event horizon, which in turn is smaller or equal than  $16\pi G^2 \omega^2$ , where  $\omega$  is as before the energy of the test particle.

Now the requirement that no black hole ruins our ability to resolve short distances

is weakened. Instead of the requirement that the energy distribution has a radius larger than the Schwarzschild radius, we only have the requirement that the area  $A$ , which encloses  $\omega$ , is large enough to prevent Penrose's condition for horizon formation:

$$A \geq 16\pi G^2 \omega^2 \quad . \quad (17)$$

The test particle interacts during a time  $\Delta x^0$  that, by the normal uncertainty principle, is larger than  $1/(2\omega)$ . Taking into account this uncertainty on the energy, one has

$$A(\Delta x^0)^2 \geq 4\pi G^2 \quad . \quad (18)$$

Now we have to make some assumption for the geometry of the object which will inevitably be a crude estimate. While an exact bound will depend on the shape of the matter distribution, we will here just be interested in obtaining a bound that depends on the three different spatial extension and is qualitatively correct. To that end, we assume the mass distribution fits into some smallest box with side-lengths  $\Delta x^1, \Delta x^2, \Delta x^3$ , which is similar to the limiting area

$$A \sim \frac{\Delta x^1 \Delta x^2 + \Delta x^1 \Delta x^3 + \Delta x^2 \Delta x^3}{\alpha^2} \quad , \quad (19)$$

where we added some constant  $\alpha$  to take into account different possible geometries. A comparison with the spherical case,  $\Delta x^i = 2R$ , fixes  $\alpha^2 = 3/\pi$  (we divert in the choice of this constant from [3], but this will not be relevant for our argument). With Eq. (18) one then obtains

$$(\Delta x^0)^2 (\Delta x^1 \Delta x^2 + \Delta x^1 \Delta x^3 + \Delta x^2 \Delta x^3) \geq 12 l_p^4 \quad . \quad (20)$$

Since

$$(\Delta x^1 + \Delta x^2 + \Delta x^3)^2 \geq \Delta x^1 \Delta x^2 + \Delta x^1 \Delta x^3 + \Delta x^2 \Delta x^3 \quad (21)$$

one also has

$$\Delta x^0 (\Delta x^1 + \Delta x^2 + \Delta x^3) \geq l_p^2 \quad . \quad (22)$$

Thus, as anticipated, taking into account that a black hole must not necessarily form if the spatial extension of a matter distribution is smaller than the Schwarzschild radius into only one direction, the uncertainty we arrive at here depends on the extension into all three directions, rather than applying separately to each.

#### 4. Minimal volume or minimal length?

Let us now compare the argument for a minimal length from section 2 with the argument against a minimal length from section 3. As mentioned earlier, that there is a bound on volumes is not the relevant statement, since this follows from the bound on the length and time intervals. Relevant is the question whether the bound (1) on the length can be violated.

First we note that in section 3 one has replaced  $\omega$  by the inverse of  $\Delta x^0$ , rather than combining with Eq. (3), but that is merely a matter of presentation and could have been done in section 2 as well.

More importantly, TV's argument is not operational. The quantities  $\Delta x''$  that they derive bounds on are not measurement outcomes. In section 2, the  $\Delta x$  that we found to be limited by the Planck length is the precision by which one can measure the position of a particle (or the presence of substructures) with help of the test particle; it has a clear physical interpretation. In section 3, the  $\Delta x^i$  are the smallest possible extensions of the test-particle (in the rest frame), which with spherical symmetry would just be the Schwarzschild radius. What is, crucially, missing in this argument is the step in which one studies the motion of the measured particle that is induced by the gravitational field of the, no longer spherically symmetric, test particle.

There is another aspect of this non-operational investigation by TV. This is the question what, fundamentally, is the test particle and how can it substantially deviate from spherical symmetry without making additional assumptions about the UV completion of the theory. We may think of a particle with a non-spherical probability distribution, one that extends to a large distance into a direction perpendicular to the  $x$ -axis, so that, by TV's argument, there is in principle nothing prohibiting us from approaching the measured particle arbitrarily. However, if the test particle does interact with the measured particle on some possible paths it can take, should we not expect its gravitational field to be that of a particle on the path rather than that of the distribution over all paths (as one would expect in semi-classical gravity)?

Of course this raises the question what is the gravitational field of a quantum superposition and what happens with it upon collapse – a question that strictly speaking is unsolved, and will only be solved by a theory of quantum gravity. Indeed, this problem is very similar to Hannah and Eppley's thought experiment [18] which purpose it was to show that the gravitational field must have quantum properties like the particle which it is created by. It seems that for TV's argument to hold, the particle's gravitational field would have to remain being spread out even after the interaction has taken place, to avoid a strong, spherically symmetric, gravitational field that delivers the distortion derived by Mead. This seems possible only if one assumes that, fundamentally, the particle is not a particle but a spatially spread-out object. At the very least, it is not clear exactly what is being measuring and how.

Finally, let us consider a concrete example for a gravitational field of the sort that TV's argument would be relevant for. The gravitational field of a line mass of finite length is the  $\gamma$ -metric and in the limit of infinite extension one obtains the Levi-Civita metric [19]. In cylinder coordinates  $r, z, \phi$ , it takes the form [20]

$$ds^2 = -dt^2 + dz^2 + dr^2 + (1 - 4G\mu)^2 r^2 d\phi^2 \quad . \quad (23)$$

The Levi-Civita metric is well-known for describing the gravitational field of a cosmic string with mass density and tension  $\mu$ . This metric has no horizon. It has



in fact the peculiar property of being flat. It just has a deficit angle of  $4\pi\mu G$ : the circumference of a circle with radius  $\tilde{r}$  is  $2\pi\tilde{r}(1 - 4\mu G)$ .

What better example would there be in support of TV's argument? If we would try to probe structures by help of such a string (at least as a limiting case) the measured particle wouldn't even notice a gravitational field of the test string. However, this argument would be short sighted. For even if the Levi-Civita metric has no horizon, and we are thus not bounded by a collapse-prohibiting requirement like in the Schwarzschild case, we still have physical considerations to take into account. If the mass density  $\mu$  exceeds  $m_{\text{Pl}}^2/2$ , the  $r, \phi$  two-space collapses to a point [20].

So we know that for physical reasons the tension of the string is bounded by the square of the Planck mass. Now if we scatter something off the string with a momentum transfer  $\omega$  that, according to the usual uncertainty principle, was large enough to test structures below the Planck-scale, we will transfer an energy at least of order  $\omega \sim m_{\text{Pl}}$  to the string, with the direction of momentum being perpendicular to the symmetry axis. This will cause the string to deform to a length that we can estimate by assuming the deformation is of triangular shape with transverse extension  $\Delta x$  and base length  $2\Delta x$ , where we have assumed that the perturbation in the string travels with the same speed that the string extends. This changes the length of the string by  $(1 - \sqrt{2})2\Delta x$ . The energy in this deformation will match the transferred energy for approximately  $\mu\Delta x = \omega$ . Thus, the extension of the string in the direction that we want to measure is

$$\Delta x \sim \frac{\omega}{\mu} \gtrsim l_{\text{Pl}}^2 \omega \quad . \quad (24)$$

Combining this with the usual uncertainty (3), we arrive again at Mead's conclusion that we cannot measure distances to better than the Planck scale.

Though the details of this argument are a little rough, this should not come as much of a surprise. The more energy we transfer to the string, the more it will deform, and the higher the tension, the less it will transform. We have no other dimensionful scale at our disposal than the Planck scale. Thus, even without knowing the details, one can tell that the uncertainty in transverse direction will have a minimum at the Planck length. And so, even though TV's argument is correct for what the gravitational field of the test particle is concerned, it does not follow from this alone that one can measure structures to arbitrary precision.

One finds a similar extension for the quantized string in string theory [21, 22]. It has been argued however in [23], that the limit on  $\Delta x$  might be avoided in string theory if one considers  $D$ -brane scattering, in which case  $\Delta x$  could be made arbitrarily small on the expense of making the interaction time  $\Delta t$  arbitrarily large, so that merely a space-time uncertainty relation

$$\Delta x \Delta t \sim l_s^2 \quad (25)$$

is valid. (Here  $l_s$  is the string scale and in general different from the Planck scale.) However, at this point we have departed quite far already from a generally applicable argument and ventured into the realms of one particular approach to quantum gravity.



## 5. Conclusion

We have shown that the argument put forward by Tomassini and Viaggiu, according to which space-time volumes are bounded but not spatial distances, is incomplete. But from our discussion we can now also see what is necessary to complete the argument: An operational way to measure structures at arbitrarily short distances that does violate the Planckian bound, presumably by use of a non-spherical geometry. It would be interesting to see exactly which assumptions about the matter content or its quantum properties are necessary, and to explore the physical consequences. Such an investigation might prove insightful for understanding the rôle of the Planck scale in different approaches to quantum gravity.

To answer the question posed in the title: Without the additional assumption that extended objects exist in the fundamental description of nature, we presently do not know of any thought experiment that would allow to measure structures to a precision better than the Planck length.

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